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REPORT NO. 812

A Least Square Attitude Solution

GEORGE R. TRIMBLE, JR.

ABERDEEN PROVING GROUND, MARYLAND

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George R. Trimble, Jr.

Project No. TB3-0838 of the Research and Development Division, Ordnance Corps

ABERDEEN PROVING GROUND, MARYLAND

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Aberdeen Proving Ground, Md.
10 June 1952

A LEAST SQUARE ATTITUDE SOLUTION

ABSTRACT

A method of determining the orientation of the axis of a missile in flight is presented. Several solutions to this problem have evolved for the case of two stations where there is no overdetermination of the parameters. If it is desirable to use data from more than two stations there is an overdetermination of the orientation of the missile axis and some adjustment procedure must be used. The solution presented herein is a least square procedure developed to solve this problem.

INTRODUCTION

Several procedures have been developed for determining the orientation of the axis of a missile in flight.[1] All of the procedures examined thus far have made use of the data from only two stations, thus there is no overdetermination of the parameters involved.

It frequently occurs that more than two stations observe the flight of the missile; consequently, the data obtained from the additional stations are of no use. If more than two stations are available, it would be possible to use one of the above mentioned procedures for each combination of two stations. The averages of the results of each combination could then be taken as the final values. This procedure greatly increases the computing time and the improvement in the final data would not justify this increase.

It becomes apparent that some adjustment procedure should be applied in order that the additional data may be utilized in case more than two stations are available. The solution presented in this paper is based upon the formulation of the problem according to the principle of least squares.

The missiles fired on large ranges are observed by more than two cameras and the least square procedure may be used. In this case, however, the cameras have a narrow field of view and the assumption that the optic axis of the camera coincides with the line of sight to the missile from the camera will be valid. This assumption will be made in the following derivations. It has been shown that the error committed in making this assumption is negligible.[2]

The missiles fired on small ranges are, in general, observed by only two, wide angle, fixed cameras. In this case the least square procedure is not applicable because there is no overdetermination and the above assumption is not valid because of the wide field of view.

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1 See Bibliography, page 14.

2 Attitude and Yaw Reductions of Projectiles in Free Flight, BRL Report No. 774, G. R. Trimble, Jr., 1951, pp. 41-43.

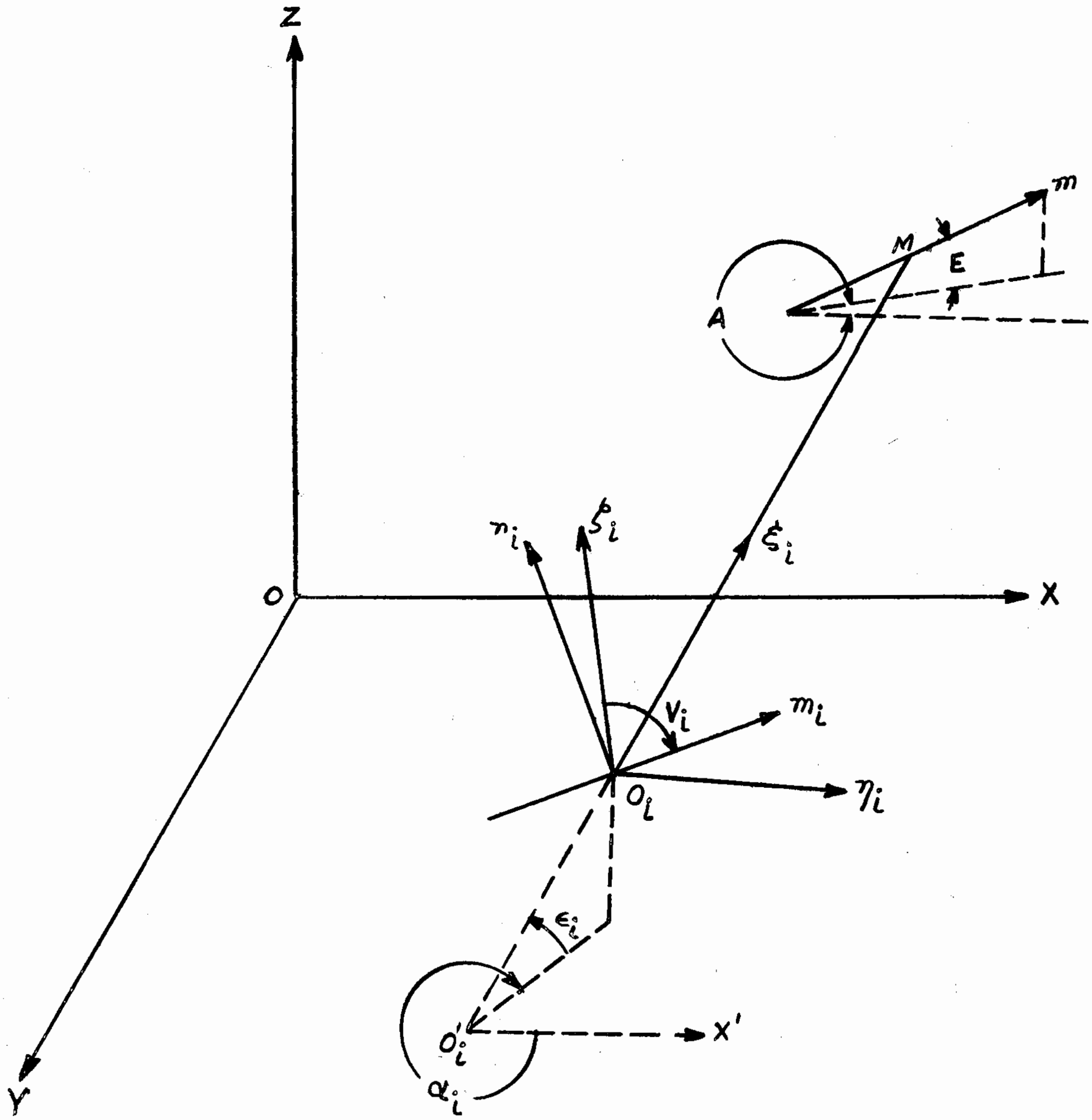

FIGURE 1

PRELIMINARY REMARKS

Before proceeding to the least square solution it is necessary to introduce several other concepts, the first of which is the coordinate system used. Any rectangular coordinate system will suffice, however, for the sake of giving some physical meaning to the concepts involved let the O-XYZ system be defined as follows (See Figure I):

(a) The origin, O, is a point on the range whose geodetic coordinates are known;

(b) The XY-plane is tangent to the earth at the origin;

(c) The X-axis is positive in the direction of the line of fire;

(d) The Y-axis is perpendicular to the X-axis positive to the right;

(e) The Z-axis is perpendicular to the XY-plane positive upward.

Let ξ, η, and ζ be unit vectors which lie along the X, Y, and Z axes respectively.

Point O_i' is the nodal point of the lens of the i^{th} camera, line $O_i'X'$ is parallel to the X-axis, line $O_i'M$ is the line of sight to some point on the missile and point O_i is a point between O_i' and M such that distance O_iO_i' is unity.

Vector ξ_i is a unit vector along the line of sight to the missile.

Vector η_i is a unit vector parallel to the XY-plane, perpendicular to ξ_i and directed to the right of ξ_i.

Vector ζ_i is a unit vector perpendicular to ξ_i and η_i, directed according to the vector right-hand thread rule.

The elevation angle, ε_i, is the angle between the line of sight and the XY-plane measured upward from the XY-plane.

The azimuth angle, α_i, is the angle between the projection of the line of sight upon the XY-plane and the X-axis, measured positive clockwise from the line $O_i'X'$, that is, from the X-axis.

In view of these definitions it is seen that

$$
(1)\begin{cases}
\xi_i = (\cos\epsilon_i \cos\alpha_i)\xi + (\cos\epsilon_i \sin\alpha_i)\eta + (\sin\epsilon_i)\zeta \\
\eta_i = (-\sin\alpha_i)\xi + (\cos\alpha_i)\eta \\
\zeta_i = (-\sin\epsilon_i \cos\alpha_i)\xi - (\sin\epsilon_i \sin\alpha_i)\eta + (\cos\epsilon_i)\zeta
\end{cases}
$$

Let m be a unit vector which lies along the missile axis directed from tail to nose.

The azimuth of the missile axis, A, is the angle between the projection of the missile axis upon the XY-plane and the X-axis, measured positive clockwise from the X-axis.

The elevation of the missile axis, E, is the angle between the missile axis and the XY-plane, measured upward from the XY-plane.

The vector m may be expressed in terms of the basis vectors ξ, η, and ζ as follows:

(2) $m = (\cos E \cos A)\xi + (\cos E \sin A)\eta + (\sin E)\zeta$

Vectors η_i and ζ_i determine a plane, commonly referred to as the "standard coordinate" plane, which is parallel to the plane of the film. Distances measured on the standard coordinate plane are simply a constant multiple of distances on the film plane. Hereafter we shall treat the standard coordinate plane as though it were the film plane since this scale change does not affect what follows.

The missile image lies along the intersection of two planes; one plane is the plane containing the missile axis and the line of sight to the missile, the other plane is the film plane. Let m_i be a unit vector which lies along the missile image.

The angle V_i is the angle between m_i and ζ_i measured positive clockwise from ζ_i. Vector ζ_i lies along the intersection of the vertical plane through the line of sight with the film plane. This intersection can be determined physically by having fiducial marks on the film. It is then possible to measure the angle V_i directly.

It is seen that

(3) $$m_i \cdot \eta_i = \sin V_i,$$

and

(4) $$m_i \cdot \zeta_i = \cos V_i.$$

Define the vector n_i by

$$n_i = (m \times \xi_i)/ \left| m \times \xi_i \right|$$

Then n_i is a unit vector normal to the plane containing m and ξ_i, consequently n_i is perpendicular to m_i, (n_i, ξ_i, m_i, η_i are thus coplanar)

Since m_i is perpendicular to both n_i and ξ_i we have

$$\begin{aligned} m_i &= \xi_i \times n_i \\ &= \xi_i \times (m \times \xi_i)/ \left| m \times \xi_i \right| \\ &= \left[(\xi_i \cdot \xi_i)\, m - (\xi_i \cdot m) \xi_i \right] / \left| m \times \xi_i \right| \\ &= \left[m - (\xi_i \cdot m) \xi_i \right] / \left| m \times \xi_i \right|, \end{aligned}$$

consequently, from equation 3 we obtain

$$\begin{aligned} \sin V_i &= \left\{ \left[m - (m \cdot \xi_i) \xi_i \right] \cdot \eta_i \right\} / \left| m \times \xi_i \right| \\ &= m \cdot \eta_i / \sqrt{1 - (m \cdot \xi_i)^2} \end{aligned}$$

and finally

(5) $\sin V_i = m \cdot \eta_i / \sin \theta_i$

where θ_i is the angle between missile axis and the line of sight to the missile.

Making use of equations 1 and 2 equation 5 becomes

(6)

$$\sin V_i = \frac{\cos E \sin (A - \alpha_i)}{\sqrt{\left[- \cos E \sin \epsilon_i \cos (A - \alpha_i) + \sin E \cos \epsilon_i \right]^2 + \left[\cos E \sin (A - \alpha_i) \right]^2}}$$

and the sign of cosine V_i as given by equation 4 may be used with the above expression to completely determine V_i and the quadrant in which it lies.

Equation 6 is convenient for computing purposes because the denominator is nothing more than $\sin \theta_i$ for which it is possible to determine a lower bound. This fact is very important when the procedure is programmed for a high speed automatic computing machine. It also gives a numerical means of determining the reliability of the V-angle itself. If $\theta = 0$, the missile axis lies along the line of sight to the missile and there would be no V-angle. For small values of θ, the missile image would be greatly foreshortened and the measured V-angle would not be very reliable. An estimate of the smallest value of θ for which the V-angles are measurable is $\theta=20^\circ$. However, to make certain that this bound is never reached a value of $\theta=10^\circ$ has been suggested as the minimum.

Mr. W. R. Clancey has developed a formula for $\tan V_i$ rather than $\sin V_i$.[1] Since most of the high speed automatic computing machines now in operation do not use a floating decimal system it would be very difficult to deal with the tangent formula on these machines because of the large valves the tangent may take. For this reason the sine formula given above is used.

[1] A Method for Calculating Rocket Attitude Data, W. R. Clancy, BRL Technical Note No. 305, August 1950.

THE LEAST SQUARE ATTITUDE SOLUTION

The theory of least squares requires that

$$(7) \quad S = \sum_{i=1}^{n} P_i \, (V_i - V_i')^2$$

$$= \sum_{i=1}^{n} P_i \left\{ \sin^{-1} \left[\frac{\cos E \sin (A-\alpha_i)}{\sqrt{\left[-\cos E \sin \epsilon_i \cos(A-\alpha_i) + \sin E \cos \epsilon_i\right]^2 + \left[\cos E \sin(A-\alpha_i)\right]^2}} \right] - V_i' \right\}^2$$

be minimized, where V_i' is the observed V-angle and P_i is a weighting factor. Differentiating S with respect to A and E and setting the derivatives equal to zero we obtain normal equations which are non-linear. These equations are very hard, if not impossible, to solve. We make use of a method described by K. Levenberg[1] which reduces the normal equations to a system of linear equations. This system of linear equations can be solved iteratively and Levenberg has shown that the iterative process converges to the "true" solution. Levenberg calls the method "Damped Least Squares".

Replace V_i by a Taylor series expansion about an approximate solution A_0, E_0, to obtain

$$V_i = V_i^o + C_{1,i}(E-E_0) + C_{2,i}(A-A_0)$$

where

$$V_i^o = \sin^{-1} \left[\frac{\cos E_0 \sin (A_0-\alpha_i)}{\sqrt{\left[-\cos E_0 \sin \epsilon_i \cos(A_0-\alpha_i) + \sin E_0 \cos \epsilon_i\right]^2 + \left[\cos E_0 \sin (A_0-\alpha_i)\right]^2}} \right],$$

$$C_{1,i} = \frac{-\cos \epsilon_i \sin(A_0-\alpha_i)}{\left[-\cos E_0 \sin \epsilon_i \cos(A_0-\alpha_i) + \sin E_0 \cos \epsilon_i\right]^2 + \left[\cos E_0 \sin(A_0-\alpha_i)\right]^2}$$

---

[1] K. Levenberg, A Method for the Solution of Certain Non-Linear Problems in Least Squares, Quarterly of Applied Mathematics, Vol. 2, No. II, 1944, pp. 164-168.

$$C_{2,i} = \frac{-\cos^2 E_0 \sin \epsilon_i + \sin E_0 \cos E_0 \cos \epsilon_i \cos (A_0 - \alpha_i)}{\left[-\cos E_0 \sin \epsilon_i \cos (A_0 - \alpha_i) + \sin E_0 \cos \epsilon_i\right]^2 + \left[\cos E_0 \sin (A_0 - \alpha_i)\right]^2}$$

and

$$C_{1,i} = \left[\frac{\partial V_i}{\partial E}\right]_{(E_0, A_0)}, \qquad C_{2,i} = \left[\frac{\partial V_i}{\partial A}\right]_{(E_0, A_0)}.$$

Denote by $\bar{S}$ the expression obtained by replacing V_i in equation 7 by $\bar{V}_i$. The procedure Levenberg describes requires the minimization of the function

(8) $F = \omega \bar{S} + (\Delta A)^2 + (\Delta E)^2$

where

$$\Delta A = A - A_0, \; \Delta E = E - E_0$$

and ω is a constant to be determined later. It is seen from equation 8 that not only are the squares of the residuals minimized but also the squares of the corrections which must be added to the approximate solution are minimized.

Differentiating F with respect to A and E and setting the derivatives equal to zero we obtain the linear normal equations

(9)

$$\begin{cases} \Delta E \left[\sum_{i=1}^{n} P_i C_{1,i}^2 + \frac{1}{\omega}\right] + \Delta A \sum_{i=1}^{n} P_i C_{1,i} C_{2,i} = \sum_{i=1}^{n} P_i C_{1,i} (V_i' - V_i^o) \\ \Delta E \sum_{i=1}^{n} P_i C_{1,i} C_{2,i} + \Delta A \left[\sum_{i=1}^{n} P_i C_{2,i}^2 + \frac{1}{\omega}\right] = \sum_{i=1}^{n} P_i C_{2,i} (V_i' - V_i^o) \end{cases}$$

It is a simple matter to solve this system of equations for ΔE and ΔA. The improved values are then given by

(10) $A = A_0 + \Delta A, \quad E = E_0 + \Delta E.$

If the values of A and E obtained in this manner are not sufficently accurate the procedure may be repeated using these improved values as the approximations. This process may be repeated as many times as required to insure accuracy commensurate with the data being analyzed.

Levenberg has shown that the best value of ω is given by

$$\omega = 1/2 \sum_{i=1}^{n} P_i (V_i' - V_i^o)^2 \Bigg/ \left\{ \left[\sum_{i=1}^{n} P_i \, c_{1,i} (V_i' - V_i^o) \right]^2 + \left[\sum_{i=1}^{n} P_i \, c_{2,i} (V_i' - V_i^o) \right]^2 \right\}$$

Trial computations indicate a value of $1/\omega = 4.5$ for the first iteration and $1/\omega = 0.1$ for the second iteration for three station reductions of Muroc bomb drops in which the final values of A and E obtained on one frame of data were used as the approximate solution for the following frame of data, and the weighting factors, P_i , were taken as unity. Experience should improve these estimates. Two iterations are sufficient to obtain an accuracy of $0^o.1$.

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A method of determining the orientation of the axis of a missile in flight is presented. Several solutions to this problem have evolved for the case of two stations where there is no over-determination of the parameters. If it is possible to use data from more than two stations there is an over-determination of the orientation of the missile axis and some

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